# Tropical scaling of a Lagrange-type linearization for matrix polynomial eigenvalue problems

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## Abstract

Let $P(z)$ be a $s \times s$ matrix polynomial of degree $d$. The *polynomial eigenvalue problem* (PEP) is to look for nonzero vectors $v$ (right eigenvectors) and corresponding eigenvalues $\lambda$ such that $P(\lambda)v = 0$.

The standard way of solving the PEP is via *linearization*, that is, by constructing a $ds \times ds$ matrix polynomial $L(z)$ of degree one such that

$$E(z)L(z)F(z) = \begin{bmatrix} P(z) & 0 \\ 0 & I_{(d-1)s} \end{bmatrix}$$

with $E(z)$ and $F(z)$ unimodular matrix polynomials. Then clearly $P(z)$ and $L(z)$ have the same eigenvalues. Many linearizations have been proposed in the literature based on the basis in which $P(z)$ is represented, e.g., degree graded bases such as the monomial basis, the Chebyshev basis, $\ldots$, or interpolation bases, such as the Lagrange polynomials. Companion linearizations are commonly used in practice for matrix polynomials expressed in the monomial basis but these are known to affect the sensitivity of eigenvalues and, when used with numerically stable eigensolvers for generalized eigenproblems, they can compute eigenpairs for $P$ with large backward errors unless the linear problem is solved several times with different scalings of the eigenvalue parameter.

The matrix polynomial $P(z)$ is uniquely determined by its values $P_i$ in $d$ points $\sigma_i$, $i = 1, 2, \ldots, d$ and its highest degree coefficient $P_d$. A Lagrange-type linearization based on this representation is

$$L(z) = \left[\begin{array}{c|cccc} P_d & \beta_1 P(\sigma_1) & \cdots & & \beta_d P(\sigma_d) \\ \hline -I_s & (z-\sigma_1)I_s & & & \\ \vdots & & & \ddots & \\ -I_n & & & & (z-\sigma_d)I_n \end{array}\right],$$

where the $\beta_i$ are the so-called barycentric weights.

We show numerically that the (well-conditioned) eigenvalues of a PEP can be computed with high relative precision using only one run of the QZ algorithm even when the eigenvalues have a large variation in magnitude. To this end a particular choice of the interpolation points $\sigma_i$, i.e., well-separated tropical roots, is taken in the Lagrange-type linearization as defined above together with an appropriate scaling. Also the QZ algorithm has to be slightly adapted such that the QZ iteration does not stop too early for eigenvalues with a large difference in magnitude. This is connected to the fact that for certain matrix pencils the QZ algorithm exhibits a structured backward error as we will illustrate by numerical experiments.